

Clausius entropy for arbitrary bifurcate null surfaces

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Abstract. Is entropy objectively “real”? Or is entropy in some sense subjective and observer-dependent? These innocent questions open a Pandora’s box of often inconclusive debate. A consensus opinion, though certainly not universally held, seems to be that the Clausius entropy (thermodynamic entropy, defined via a Clausius relation $dS = dQ/T$) should be objectively real, but that the ontological status of the statistical entropy (the Shannon or von Neumann entropy) is much more ambiguous, and much more likely to be observer-dependent. This question is particularly pressing when it comes to understanding Bekenstein entropy (black hole entropy). To perhaps further add to the confusion, we shall argue that even Clausius entropy can often be observer-dependent. We shall demonstrate that one can meaningfully assign a notion of Clausius entropy to the matter crossing arbitrary bifurcate null surfaces — effectively defining a “virtual Clausius entropy” for arbitrary “virtual causal horizons”.

Keywords: Entropy, Clausius entropy, thermodynamic entropy, statistical entropy, Bekenstein entropy, bifurcate null surfaces, causal horizons. arXiv:1303.3185 [gr-qc]

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1. Introduction

The ontological status of entropy continues to generate much heated and inconclusive debate. Key questions are: Is entropy objectively “real”? Or is entropy in some sense subjective and observer-dependent? Part of the issue is that there are many different notions of entropy, and the extent to which they are equivalent is less than clear. At a minimum, one might wish to consider:

- Clausius entropy ($dS = dQ/T$); often called thermodynamic entropy [1].
- Bekenstein entropy; black hole entropy [2].
- Statistical entropy; (Shannon [3], von Neumann [4], or entanglement entropy).

Other notions of entropy include Gibbs entropy, Boltzmann entropy, Srednicki entropy, Kolmogorov–Sinai entropy, the Tsallis and Renyi entropies. Several attempts have been carried out in order to reconcile these different definitions [5, 6, 7, 8, 9, 10, 11] and also to separate out equilibrium and non-equilibrium notions of entropy [12]. For general background see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Typically, the Clausius entropy is viewed as the most objectively real of these entropies, (and will be the central focus of this article), but even there the situation is extremely subtle. (See for instance the discussion provided by Padmanabhan [24, 25, 26].) These ontological issues are central to Jacobson’s “thermodynamic” derivation of the Einstein equations [27], where one part of the argument is based on an entanglement entropy interpretation of (a variant of) the Bekenstein entropy, and another part of the argument is based on (a variant of) the Clausius entropy applied to “local Rindler horizons”. Instead of addressing Jacobson’s “thermodynamic” derivation directly we shall in the current article address a more modest goal: To what extent can a Clausius-type notion of entropy be associated with matter crossing *arbitrary* bifurcate null surfaces?

The construction presented in the current article is considerably more general than Jacobson’s construction, which associated Clausius entropy only with local Rindler horizons. We shall soon see that while the bifurcate nature of the local Rindler horizon is essential to the construction, other Rindler-specific features can easily be discarded. Any null surface can be viewed as an observer-dependent causal horizon, a “virtual” causal horizon — and our construction can be viewed as providing a notion of virtual Clausius entropy for matter crossing *arbitrary* bifurcate virtual causal horizons — thereby undermining the ontological reality of Clausius entropy; modulo some technical assumptions that we shall be very careful to make explicitly clear.

Building on this construction, in future work we plan to address the issue of the extent to which Bekenstein and Clausius entropies can universally be inter-related. Specifically: For which subset of causal horizons (virtual or otherwise) *should* they be inter-related? Under what situations should these concepts carefully be kept distinct?

2. Flat Minkowski spacetime

To start the calculation is best to work in flat Minkowski spacetime. That is, for now we are working in the framework of SR, not GR. To get a handle on the notion of heat flux $\mathfrak{d}Q$ it is convenient to start with a infinitesimal segment of timelike hypersurface, (ruled by a congruence of future-pointing timelike vectors V^a , with outward spacelike normal n^a , and with hypersurface area element $\mathfrak{d}^3\Sigma$), and define:

$$\mathfrak{d}Q = T_{ab} V^a n^b \mathfrak{d}^3\Sigma. \quad (1)$$

For finite segments of hypersurface we set

$$\mathfrak{d}Q = \int T_{ab} V^a n^b \mathfrak{d}^3\Sigma. \quad (2)$$

There would be universal agreement that this quantity defines the net energy flux across the segment of timelike hyper-surface, but perhaps less agreement that this energy flux can be equated with a heat flux. Following Jacobson [27], let us nevertheless accept this definition for the sake of argument and see where this identification leads. Note that due to the $(-; + + +)$ signature of spacetime this is, perhaps counter-intuitively, the *inwards* flux of energy.

We can then consider a sequence of timelike hypersurfaces, and construct an appropriate null limit. From the way this limit is set up it will soon be clear that we cannot deal with completely arbitrary null surfaces — the construction intrinsically is set up so that the limit automatically yields bifurcate null surfaces. One of the advantages of Minkowski space is that it is possible to develop some *exact* results, many of which will even hold *globally*. We will subsequently invoke local flatness to extract more limited approximate results in curved spacetimes; approximate results which nevertheless hold up to an explicitly controlled level of accuracy in the vicinity of the bifurcation 2-surface. Ultimately we shall develop a construction valid in arbitrary curved spacetimes.

A key physics step in the computation is to invoke the Unruh effect (acceleration radiation). This is a flat-space SR QFT result whereby an accelerated observer will, (when the QFT is in its usual SR ground state), detect a thermal bath of quantum excitations with a temperature [28]:

$$k_B T = \frac{\hbar a}{2\pi}. \quad (3)$$

We shall use the Unruh effect to *define* the differential Clausius entropy for the matter crossing any timelike hypersurface segment swept out by timelike observers of 4-acceleration a by:

$$\mathfrak{d}S = \frac{\mathfrak{d}Q}{T} = \frac{2\pi k_B}{\hbar a} \mathfrak{d}Q = \frac{2\pi k_B}{\hbar a} T_{ab} V^a n^b \mathfrak{d}^3\Sigma. \quad (4)$$

For a finite segment of hypersurface we could in principle allow the acceleration a to vary from place to place and would then have

$$\mathfrak{d}S = \frac{2\pi k_B}{\hbar} \int \frac{T_{ab}}{a} V^a n^b \mathfrak{d}^3\Sigma. \quad (5)$$

In all explicit calculations below the hypersurfaces will be set up in such a manner that the acceleration a is a constant over the hypersurface so that

$$dS = \frac{2\pi k_B}{\hbar a} \int T_{ab} V^a n^b d^3\Sigma. \quad (6)$$

A tricky point is that the T_{ab} being used here is purely classical, whereas the Unruh temperature is associated with quantum fluctuations in the quantum ground state. In the presence of excitations above the quantum ground state it can be argued that the Unruh effect provides a lower bound on the physical temperature [29, 30, 31, 32], and so an upper bound on $|dQ|/T$. Furthermore the dS defined above is a “virtual” quantity; there is no actual need for the timelike observers to be objectively real and physically present — the dS defined above is what *would be seen* by an imaginary swarm of timelike observers skimming along the timelike hypersurface.

In view of these issues, (identification of the heat flux, identification of the temperature, virtual status of the quantity dS), some may refuse to call the quantity dS a Clausius entropy, and prefer to introduce yet another notion — perhaps “Jacobson entropy” might be appropriate? Be that as it may, provided one accepts this definition, and we hope the reader will agree this is a very plausible and physically interesting object to calculate, most of the technical computations of this article boil down to taking appropriate limits as the acceleration a tends to infinity and the timelike surface becomes null.

2.1. Rindler wedges

Let us pick an arbitrary spacelike 2-plane in Minkowski space and choose coordinates so that this plane is

$$x^a(x, y) = (0; x, y, 0). \quad (7)$$

Now add past and future light sheets, for convenience in the $+z$ direction. The resulting bifurcate null surface is

$$x^a(t, x, y) = (t; x, y, |t|). \quad (8)$$

The two null 3-d half-planes are joined by the spacelike bifurcation 2-plane at $t = 0$. Now pick a sheet of hyperbolic timelike observers “close” to that null surface:

$$x^a(\tau; x, y) = \left(\frac{1}{a} \sinh(a\tau); x, y, \frac{1}{a} \cosh(a\tau) \right). \quad (9)$$

(Eventually we will want to take $a \rightarrow \infty$.) These observers have 4-velocity

$$V^a(\tau; x, y) = (\cosh(a\tau); 0, 0, \sinh(a\tau)); \quad ||V|| = 1; \quad (10)$$

and 4-acceleration

$$A^a(\tau; x, y) = a (\sinh(a\tau); 0, 0, \cosh(a\tau)); \quad ||A|| = a; \quad (11)$$

while the hyperbolic timelike sheet they sweep out has 4-normal

$$n^a(\tau; x, y) = (\sinh(a\tau); 0, 0, \cosh(a\tau)); \quad ||n|| = 1. \quad (12)$$

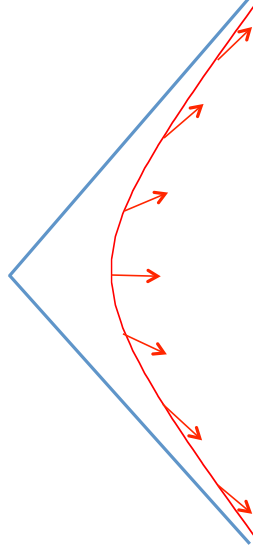


Figure 1. Rindler wedge with hyperbolic timelike observer and 4-normal. Note that the 4-normal asymptotes to the 4-tangent at extremely late and extremely early proper times.

Note that on the time-like shell we have the restriction

$$T_{ab}(x^a) \rightarrow T_{ab}(\tau, x, y). \quad (13)$$

Then, setting $d^2\mathcal{A} = dx dy$ we have

$$dQ = \int T_{ab} V^a n^b d\tau d^2\mathcal{A}, \quad (14)$$

and so

$$\frac{dQ}{d\tau} = \int T_{ab} V^a n^b d^2\mathcal{A}, \quad (15)$$

whence

$$\frac{dQ}{dt} = \int T_{ab} V^a n^b \frac{d\tau}{dt} d^2\mathcal{A}. \quad (16)$$

With current conventions this is the *inwards* flux of matter crossing the time-like shell.

Now compute:

$$\frac{dQ}{dt} = \int \{ [T_{00} + T_{33}] \sinh a\tau \cosh a\tau + T_{03} [\sinh^2 a\tau + \cosh^2 a\tau] \} \frac{d\tau}{dt} d^2\mathcal{A}. \quad (17)$$

Substituting

$$\sinh(a\tau) = at; \quad \cosh(a\tau) = \sqrt{1 + (at)^2}, \quad (18)$$

and

$$\cosh(a\tau) d\tau = dt; \quad \frac{d\tau}{dt} = \frac{1}{\sqrt{1 + (at)^2}}, \quad (19)$$

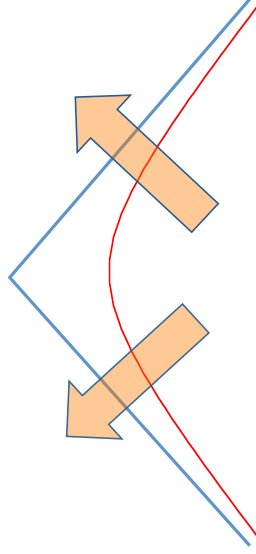


Figure 2. Formal direction of the heat flux $\mathrm{d}Q$.

we see that we have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \left\{ [T_{00} + T_{33}]at\sqrt{1 + (at)^2} + 2T_{03}[2 + (at)^2] \right\} \frac{1}{\sqrt{1 + (at)^2}} \mathrm{d}^2\mathcal{A}, \quad (20)$$

thereby implying

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = a \int \left\{ [T_{00} + T_{33}]t + 2T_{03} \left[\frac{2 + (at)^2}{a\sqrt{1 + (at)^2}} \right] \right\} \mathrm{d}^2\mathcal{A}. \quad (21)$$

Therefore, invoking the Unruh effect relating the temperature T to magnitude of the 4-acceleration a , and explicitly using $k_B T = \hbar a / (2\pi)$, we now have

$$\frac{\mathrm{d}Q/\mathrm{d}t}{T} = \frac{2\pi k_B}{\hbar} \frac{\mathrm{d}Q/\mathrm{d}t}{a} = \frac{2\pi k_B}{\hbar} \int \left\{ [T_{00} + T_{33}]t + 2T_{03} \left[\frac{2 + (at)^2}{a\sqrt{1 + (at)^2}} \right] \right\} \mathrm{d}^2\mathcal{A}. \quad (22)$$

The key point is that this now has a well-defined limit as $a \rightarrow \infty$. Indeed

$$\begin{aligned} \frac{\mathrm{d}Q/\mathrm{d}t}{T} &\rightarrow \frac{2\pi k_B}{\hbar} \int \{ [T_{00} + T_{33}] t + 2T_{03} |t| \} \mathrm{d}^2\mathcal{A} \\ &= \frac{2\pi k_B}{\hbar} t \int \{ [T_{00} + T_{33}] + 2T_{03} \operatorname{sign}(t) \} \mathrm{d}^2\mathcal{A}. \end{aligned} \quad (23)$$

Note that the 2-d integral is to be evaluated on the transverse 2-plane (the x - y plane) at time t . Now defining the null vectors

$$k_{\pm}^a = (1; 0, 0, \operatorname{sign}(t)), \quad (24)$$

which are the null normals on the two segments of the null surface, we have

$$\frac{\mathrm{d}Q/\mathrm{d}t}{T} = \frac{2\pi k_B}{\hbar} t \int \{ T_{ab} k_{\pm}^a k_{\pm}^b \} \mathrm{d}^2\mathcal{A}. \quad (25)$$

It is important to realise this is an *exact* result, valid *globally* for all time. Under normal circumstances the null energy condition [NEC] is satisfied [33], then $\{T_{ab} k_{\pm}^a k_{\pm}^b\} \geq 0$ and the flux is inwards and positive for $t > 0$. On the other hand, the inward flux is negative for $t < 0$, indicating that it should be reinterpreted as a positive outward flux.

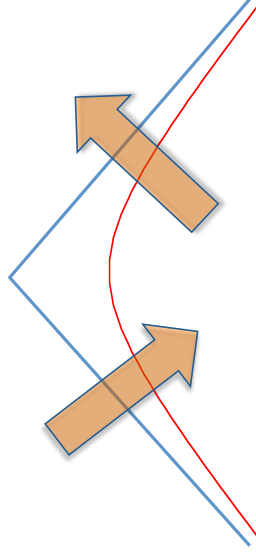


Figure 3. Physical direction of the heat flux dQ assuming the NEC holds.

The only potentially naively unexpected part of this result is that it is explicitly linear in t . Technically that feature can ultimately be traced back to three facts: That the location of the bifurcation 2-surface picks out a particular preferred origin for the time coordinate, that symmetry enforces the flux to be zero at the bifurcation 2-surface, and furthermore that

$$\frac{1}{a} \frac{d\tau}{dt} = \frac{\sqrt{1 + (at)^2}}{a} = \sqrt{\frac{1}{a^2} + t^2} \rightarrow |t|. \quad (26)$$

Note that the limiting procedure is explicitly essential to get the explicit factor of t above. Also, the limiting procedure (starting from a timelike sheet) is needed for us to be able to invoke the Unruh effect — since the Unruh effect really makes sense only for timelike observers. We can now unambiguously write down Clausius entropy *differences* for arbitrary times on the Rindler sheets:

$$\Delta S_{\text{Clausius}}(t_1, t_2) = \frac{2\pi k_B}{\hbar} \int_{t_1}^{t_2} \bar{t} \int T_{ab}(\bar{t}, x, y) k_{\pm}^a k_{\pm}^b d^2 \mathcal{A} d\bar{t}. \quad (27)$$

Perhaps more tellingly we can (in Minkowski space) usefully define the Clausius entropy of the Rindler wedge at time t as:

$$S_{\text{Clausius}}(t) = S_B + \frac{2\pi k_B}{\hbar} \int_0^t \bar{t} \int T_{ab}(\bar{t}, x, y) k_{\pm}^a k_{\pm}^b d^2 \mathcal{A} d\bar{t}, \quad (28)$$

where S_B is the entropy to be associated with the bifurcation 2-plane itself, a quantity which is *not* constrained by the current argument. Again we emphasise that this is an *exact* result, valid *globally* for all time.

2.2. Causal null cones

It is now easy to see that the construction above is not limited to Rindler wedges and flat null sheets. (Which is why we spent so much time on the explicit calculation above.) Consider for instance causal null cones defined as follows: Choose a spacelike 2-sphere of radius r_0 , with attached light cones expanding to both future and past. Adopt spherical polar coordinates so that the spacelike 2-sphere is

$$x^a(\theta, \phi) = (0; r_0, \theta, \phi), \quad (29)$$

while the null surface is:

$$x^a(t; \theta, \phi) = (t; r_0 + |t|, \theta, \phi). \quad (30)$$

Now pick a spherical sheet of timelike observers

$$x^a(\tau, \theta, \phi) = \left(\frac{1}{a} \sinh(a\tau); r_0 + \frac{1}{a} \cosh(a\tau), \theta, \phi \right), \quad (31)$$

now with 4-velocity

$$V^a(\tau, \theta, \phi) = (\cosh(a\tau); \sinh(a\tau), 0, 0); \quad ||V|| = 1, \quad (32)$$

and 4-acceleration

$$A^a(\tau, \theta, \phi) = a (\sinh(a\tau); \cosh(a\tau), 0, 0); \quad ||A|| = a, \quad (33)$$

and 4-normal

$$n^a(\tau, \theta, \phi) = (\sinh(a\tau); \cosh(a\tau), 0, 0); \quad ||n|| = 1. \quad (34)$$

Then

$$\mathrm{d}Q = \int \left(r_0 + \frac{1}{a} \cosh(a\tau) \right)^2 T_{ab} V^a n^b \mathrm{d}\tau \mathrm{d}^2\Omega. \quad (35)$$

Again, note that this is an *inwards* entropy flux.

Now compute:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= \int \left(r_0 + \frac{1}{a} \cosh(a\tau) \right)^2 \\ &\quad \times \{ [T_{00} + T_{11}] \sinh a\tau \cosh a\tau + T_{01} [\sinh^2 a\tau + \cosh^2 a\tau] \} \frac{\mathrm{d}\tau}{\mathrm{d}t} \mathrm{d}^2\Omega. \end{aligned} \quad (36)$$

Strategically it is now best to substitute

$$\sinh(a\tau) = at; \quad \cosh(a\tau) = \sqrt{1 + (at)^2}. \quad (37)$$

and

$$\cosh(a\tau) \mathrm{d}\tau = \mathrm{d}t; \quad \mathrm{d}\tau = \frac{\mathrm{d}t}{\sqrt{1 + (at)^2}}. \quad (38)$$

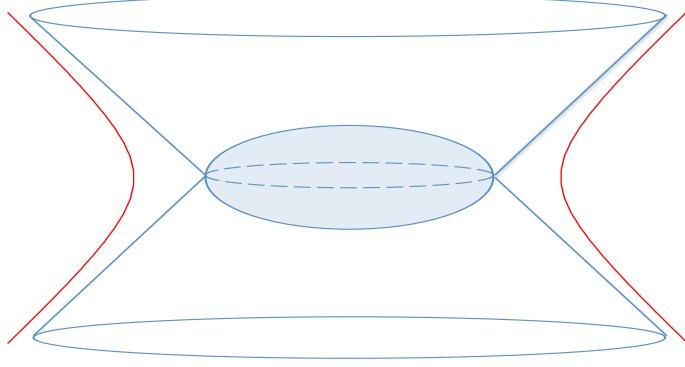


Figure 4. Bifurcate double null cone based on a spherical bifurcation 2-surface. Typical timelike observers indicated by red lines.

Then

$$\frac{dQ}{dt} = \int \left(r_0 + \frac{\sqrt{1 + (at)^2}}{a} \right)^2 \times \left\{ [T_{00} + T_{11}]at\sqrt{1 + (at)^2} + T_{01}[1 + 2(at)^2] \right\} \frac{1}{\sqrt{1 + (at)^2}} d^2\Omega, \quad (39)$$

and so

$$\frac{dQ}{dt} = \int \left(r_0 + \frac{\sqrt{1 + (at)^2}}{a} \right)^2 \left\{ [T_{00} + T_{11}]at + T_{01} \frac{[1 + 2(at)^2]}{\sqrt{1 + (at)^2}} \right\} d^2\Omega. \quad (40)$$

Therefore, invoking the Unruh effect,

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} \frac{dQ/dt}{a} \quad (41)$$

$$= \frac{2\pi k_B}{\hbar} \int \left(r_0 + \frac{\sqrt{1 + (at)^2}}{a} \right)^2 \left\{ [T_{00} + T_{11}]t + T_{01} \frac{[1 + 2(at)^2]}{a\sqrt{1 + (at)^2}} \right\} d^2\Omega. \quad (42)$$

This now has a well-defined limit as $a \rightarrow \infty$. Indeed we have the *exact* result

$$\frac{dQ/dt}{T} \rightarrow \frac{2\pi k_B}{\hbar} \int (r_0 + |t|)^2 \{ [T_{00} + T_{11}]t + 2T_{01}|t| \} d^2\Omega \quad (43)$$

$$= \frac{2\pi k_B}{\hbar} (r_0 + |t|)^2 t \int \{ T_{ab} k_{\pm}^a k_{\pm}^b \} d^2\Omega. \quad (44)$$

Here

$$k_{\pm}^a = (1; \text{sign}(t), 0, 0), \quad (45)$$

and the angular integral is to be carried out over the 2-sphere at time t . Pulling the factor $(r_0 + |t|)^2$ inside the integral we obtain the *exact* result

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} t \int \{ T_{ab} k_{\pm}^a k_{\pm}^b \} d^2\mathcal{A}. \quad (46)$$

The integral is now over the area of the 2-sphere at time t . Formally the final result is completely equivalent to that obtained for the Rindler wedge, even though various

intermediate steps were somewhat different. This observation is particularly important, in that it will now allow us to greatly extend the range of validity of our previous result. In particular for any causal null cone (light cone) we now have

$$S_{\text{Clausius}}(t) = S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \int_0^t \bar{t} \int T_{ab}(\bar{t}, \mathbf{x}(\theta, \phi, \bar{t})) k_{\pm}^a k_{\pm}^b d^2 \mathcal{A} d\bar{t}, \quad (47)$$

Here $S_{\mathcal{B}}$ is now the Clausius entropy to be associated with the bifurcation 2-sphere of radius r_0 located at $t = 0$.

2.3. Convex-base null conoids

Consider now an arbitrary convex spacelike 2-surface. Choose Cartesian coordinates x^a in Minkowski space, and generic coordinates ξ^i on the 2-surface. Then we can write

$$x^a(\xi^i) = (0; \mathbf{x}(\xi^i)). \quad (48)$$

Because the surface is convex, its outward pointing normals $\mathbf{n}(\xi)$ never intersect, so we can attach outward pointing past and future light rays to each point on the surface, and in turn these light rays will never intersect — so they define null surfaces. Then for these null sheets

$$x^a(t; \xi^i) = (t; \mathbf{x}(\xi^i) + |t| \mathbf{n}(\xi^i)). \quad (49)$$

The resulting null conoids intersect at the original spacelike 2-surface, which is therefore a bifurcation 2-surface. Along each one of these normal directions we can now simply copy the calculation above — which is why we put the effort into an explicit calculation for those simple cases. We again see

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} t \int \{T_{ab} k_{\pm}^a k_{\pm}^b\} d^2 \mathcal{A}. \quad (50)$$

The integral is over the cross-sectional area of the conoid at time t . The result is again *exact* and valid *globally* for all time. We again see

$$S_{\text{Clausius}}(t) = S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \int_0^t \bar{t} \int T_{ab}(\bar{t}, \mathbf{x}(\xi, t)) k_{\pm}^a k_{\pm}^b d^2 \mathcal{A} d\bar{t}, \quad (51)$$

2.4. Causal diamonds

To understand what happens if the bifurcation 2-surface is concave, (even partially concave), it is best to start with the highly-symmetric causal diamond configuration. Choose a spacelike 2-sphere of radius r_0 , but now with attached light cones *contracting* to both future and past. The null surface is now

$$x^a(t; \theta, \phi) = (t; r_0 - |t|, \theta, \phi). \quad (52)$$

The null curves generating the null surface now all collide at two points at $t_{\text{collision}} = \pm r_0$. A suitable class of timelike observers is now

$$x^a(\tau, \theta, \phi) = \left(\frac{1}{a} \sinh(a\tau); r_0 - \frac{1}{a} \cosh(a\tau), \theta, \phi \right), \quad (53)$$

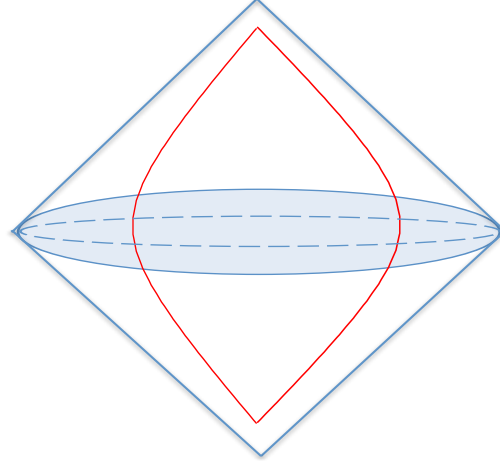


Figure 5. Causal diamond configuration.
Typical timelike observers indicated by red lines.
Note timelike observers now collide at finite time.

with the timelike observers colliding at

$$\tau_{\text{collision}} = \pm \frac{1}{a} \cosh^{-1}(r_0 a), \quad (54)$$

corresponding to

$$t_{\text{collision}} = \pm \sqrt{r_0^2 - \frac{1}{a^2}}. \quad (55)$$

As long as we restrict attention to the finite interval where these timelike curves do not intersect then the previous computation for causal null cones can be carried over, and taking the appropriate limit we see that in the finite interval $t \in (-r_0, r_0)$ we still have

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} t \int \{T_{ab} k_{\pm}^a k_{\pm}^b\} d^2 \mathcal{A}. \quad (56)$$

The only minor quirk is that the timelike observers now reside *inside* the null surface, and that the timelike observers have 4-normal

$$n^a(\tau, \theta, \phi) = (\sinh(a\tau); -\cosh(a\tau), 0, 0); \quad ||n|| = 1. \quad (57)$$

This implies one is now calculating an *outward* entropy flux. If one assumes the NEC this corresponds to a positive outwards flux for $t > 0$ and a negative outwards (positive inwards) flux for $t < 0$.

2.5. Generic null conoids

Consider arbitrary null conoids in flat Minkowski space. We start with some arbitrary 2-surface at time $t = 0$,

$$x^a(\xi^i) = (0; \mathbf{x}(\xi^i)), \quad (58)$$

but now with no constraint on the convexity of the 2-surface. All the real work has already been done — the only obstruction comes from intersecting null normals. We see

that over some finite interval $t \in (-t_*, t_*)$, where t_* is determined by the time of first intersection of the null normals, we still have the *exact* result

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} t \int \{T_{ab} k_{\pm}^a k_{\pm}^b\} d^2 \mathcal{A}. \quad (59)$$

Consequently, for the Clausius entropy, at as long as $t \in (-t_*, t_*)$, we still have the *exact* result

$$S_{\text{Clausius}}(t) = S_B + \frac{2\pi k_B}{\hbar} \int_0^t \bar{t} \int T_{ab}(\bar{t}, \mathbf{x}(\xi, \bar{t})) k_{\pm}^a k_{\pm}^b d^2 \mathcal{A} d\bar{t}. \quad (60)$$

While the discussion started out with a straightforward computation for Rindler horizons, we have now bootstrapped it to a large class of bifurcate null surfaces (still in Minkowski space) — the only limitation at this stage is that there be an inertial frame in which the bifurcation surface can be chosen to lie on the hyperplane $t = 0$.

2.6. Generic bifurcate null surfaces

As a penultimate step, we are now ready to address the situation for generic bifurcate null sheets in flat Minkowski space. We start with some completely arbitrary spacelike 2-surface,

$$x^a(\xi^i) = (t_0(\xi^i); \mathbf{x}(\xi^i)), \quad (61)$$

but now with no constraint on the convexity of the 2-surface, nor with any constraint that the 2-surface be contained in a hyperplane. Picking normals $\mathbf{n}(\xi)$ to the spatial part of this 2-surface, so we can attach outward pointing past and future light rays to each point on the surface — thereby defining null surfaces. Then for these null sheets one convenient parameterization is

$$x^a(t; \xi^i) = (t_0(\xi^i) + t; \mathbf{x}(\xi^i) + |t| \mathbf{n}(\xi^i)) = x^a(\xi^i) + t k_{\pm}^a(\xi^i). \quad (62)$$

Again, all the real work has already been done — the only significant obstruction comes from intersecting null normals. We see that over some finite interval $t \in (-t_*, t_*)$, where t_* is determined by the time of first intersection of the null normals, we still have the *exact* result

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} t \int \{T_{ab}(x(\xi) + t k_{\pm}(\xi)) k_{\pm}^a k_{\pm}^b\} d^2 \mathcal{A}. \quad (63)$$

One subtlety is that this is not precisely dQ/dt “at physical time t ”; rather this dQ/dt is obtained by propagating the bifurcation surface \mathcal{B} forward by time t in some arbitrarily chosen rest frame and calculating the flux as a function of this evolution parameter. Consequently, for the Clausius entropy we can still write down an *exact* result

$$S_{\text{Clausius}}(t) = S_B + \frac{2\pi k_B}{\hbar} \int_0^t \bar{t} \int T_{ab}(x(\xi) + \bar{t} k_{\pm}(\xi)) k_{\pm}^a k_{\pm}^b d^2 \mathcal{A} d\bar{t}. \quad (64)$$

As a final step we note that we could independently pick an affine parameter λ on each null generator and write

$$x^a(\lambda; \xi^i) = x^a(\xi^i) + \lambda k_{\pm}^a(\xi^i); \quad k_{\pm}^a(\xi^i) = \frac{dx^a(\lambda, \xi^i)}{d\lambda}. \quad (65)$$

Then taking \mathcal{S} to be any spacelike cross-section of the bifurcate null surface we have

$$S_{\text{Clausius}}(\mathcal{S}) = S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \int_{\mathcal{B}}^{\mathcal{S}} \lambda T_{ab}(x(\xi) + \lambda k_{\pm}(\xi)) k_{\pm}^a k_{\pm}^b d^2\mathcal{A} d\lambda. \quad (66)$$

Here the integral now runs over the entire null surface between \mathcal{B} and \mathcal{S} . Note that the construction is manifestly independent of the way the affine parameter is normalized. We emphasize that while the discussion started out with a straightforward computation for Rindler horizons, we have now bootstrapped it to essentially arbitrary bifurcate null surfaces (still in Minkowski space). We shall now perform a consistency check on the reasonableness of the construction, and then generalize the construction to curved spacetimes.

3. Compatibility with the Bekenstein bound

Let us now check our proposal for the Clausius entropy for compatibility with the Bekenstein bound [34]:

$$S \leq k_B \frac{2\pi M R}{\hbar}. \quad (67)$$

This inequality was argued by Bekenstein to apply to weakly bound and weakly interacting systems. Since our Clausius notion of entropy is at this stage purely a Minkowski space result, the system is certainly weakly bound. But how are we to take this quantity,

$$S_{\text{Clausius}}(t) = S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \int_0^t \bar{t} \int T_{ab}(\bar{t}, \mathbf{x}(t)) k_{\pm}^a k_{\pm}^b d^2\mathcal{A} d\bar{t}, \quad (68)$$

and relate it to Bekenstein's bound? Certainly some extra assumptions will be required. (Such as, which bifurcate null surfaces will we consider?)

Let us first choose the bifurcation surface to be a single point, and the null surface to be its future light cone. When the bifurcation surface is a single point it is plausible to set $S_{\mathcal{B}} \rightarrow 0$. For simplicity, let us first take the stress-energy to be that of a spherically symmetric perfect fluid, then

$$S_{\text{Clausius}}(t) = \frac{2\pi k_B}{\hbar} 4\pi \int_0^t \bar{t}^3 (\rho + p) d\bar{t}. \quad (69)$$

Further note that $t \rightarrow R$, the radius of the light-sphere at time t . (We have set $c \rightarrow 1$.) For the specific case of a constant density fluid we then have

$$S_{\text{Clausius}}(R) = \frac{2\pi k_B}{\hbar} 4\pi(\rho + p) \frac{R^4}{4}. \quad (70)$$

But the Bekenstein bound is asserted to apply to weakly interacting systems, so it is for current purposes acceptable to take $p \in (0, \rho/3)$. This is equivalent to the ‘‘trace energy condition’’. The TEC is one of the oldest of the classical energy conditions, which was subsequently abandoned as fundamental physics, though it is certainly a useful

characterization for weakly interacting matter [33]. Under these conditions $\rho + p < \frac{4}{3}\rho$, and so we have

$$S_{\text{Clausius}}(R) < \frac{2\pi k_B}{\hbar} \frac{4\pi\rho R^3}{3} R = k_B \frac{2\pi MR}{\hbar}, \quad (71)$$

as required. Consequently the notion of Clausius entropy defined in this article is indeed compatible with the Bekenstein bound. This gives us additional confidence that the construction developed above is physically interesting.

If the density and pressure are not constant (but are at least spherically symmetric) a minor variant of the above argument considers the quantity

$$X = 4\pi \int_0^R r^3(\rho + p) dr < 4\pi \int_0^R r^3(4\rho/3) dr = \int_0^R r dm(r) + \frac{1}{3} \int_0^R r dm(r). \quad (72)$$

But then by integration by parts

$$X < MR - \int_0^R \left[m(r)dr - \frac{1}{3} r dm(r) \right] = MR + \frac{1}{3} \int_0^R r^4 \frac{d[m(r)/r^3]}{dr} dr. \quad (73)$$

If we now assume the average density is decreasing as one moves outwards, then $d[m(r)/r^3]/dr < 0$, and the last term is negative. This falloff condition on the average density is one of the specific conditions Chandrasekhar uses in his investigations of non-relativistic stellar structure [35]. Then $X < MR$, and we again see that our construction for the Clausius entropy is at least compatible with Bekenstein's bound for weakly interacting systems.

4. Curved spacetime

Now that we have carried out this exact calculation for flat Minkowski space, and checked for compatibility with wider notions of what we expect entropy to be, the generalization to curved spacetime is straightforward

4.1. Near the bifurcation 2-surface

First, consider an approximate calculation for curved spacetime in the vicinity of the bifurcation 2-surface. Pick a bifurcate null surface in some curved spacetime. Pick a point on that bifurcation 2-surface. In the vicinity of that point adopt Gaussian normal coordinates x^a so that

$$g_{ab} = \eta_{ab} + \mathcal{O}([x^a]^2). \quad (74)$$

Then to the appropriate level of accuracy the null curves emanating from this point on the bifurcation 2-surface can be represented by

$$x^a(t) = (t; 0, 0, |t|) + \mathcal{O}(t^2). \quad (75)$$

An appropriate timelike observer is

$$x^a(\tau) = \left(\frac{1}{a} \sinh(a\tau); 0, 0, \frac{1}{a} \cosh(a\tau) \right) + \mathcal{O}(\tau^2). \quad (76)$$

Equivalently

$$x^a(t) = \left(t; 0, 0, \sqrt{t^2 + \frac{1}{a^2}} \right) + \mathcal{O}(t^2). \quad (77)$$

Differentiating, the 4-velocity and 4-normal are determined up to terms of $\mathcal{O}(t)$, and the 4-acceleration up to terms of $\mathcal{O}(1)$. Furthermore, note that $T_{ab}(t) = T_{ab}(0) + \mathcal{O}(t)$.

Finally, although the timelike observers are no longer exactly hyperbolic for all time, there is an adiabatic argument [36] demonstrating that the Unruh effect will still hold adiabatically as long as the region over which the motion is close to hyperbolic (the size of this region being determined by the spacetime curvature) is large compared to the distance scale $1/a$. (Similarly, in a black hole situation the existence of the Hawking effect is not dependent on the presence of an exact stationary [event] horizon, an approximate horizon satisfying a suitable adiabaticity condition is quite sufficient for the emission of a Planckian spectrum of Hawking photons [37, 38, 39, 40].)

Inserting all this into the computation, and taking the limit $a \rightarrow \infty$, we now get the approximate result

$$\frac{dQ/dt}{T} = \frac{2\pi k_B}{\hbar} t \int_{\mathcal{B}} \{T_{ab}(0) k_{\pm}^a k_{\pm}^b\} d^2\mathcal{A} + \mathcal{O}(t^2), \quad (78)$$

where to the relevant level of approximation the integral now runs over the bifurcation 2-surface \mathcal{B} . A subtlety here is that the Gaussian normal coordinate construction implies that one is free to choose the t coordinate independently on each null generator of the bifurcate null surface. This is equivalent to the ability to choose an arbitrary affine parameter λ for each null generator and to make this more explicit we can write

$$\frac{dQ/d\lambda}{T} = \frac{2\pi k_B}{\hbar} \lambda \int_{\mathcal{B}} \{T_{ab}(0) k_{\pm}^a k_{\pm}^b\} d^2\mathcal{A} + \mathcal{O}(\lambda^2). \quad (79)$$

If we restrict attention to “locally Rindler” bifurcate null surfaces then this expression is one of the key steps in Jacobson’s thermodynamic derivation of the Einstein equations [27] — but it is now clear from the present discussion that at least this aspect of Jacobson’s argument is much more general, applying to essentially arbitrary bifurcate null surfaces. Note that this is the inward entropy flux. For positive t and matter satisfying the NEC the flux is positive inwards. The sign flip for negative t indicates the entropy flow is then positive outwards. Consequently, for the Clausius entropy we now have

$$S_{\text{Clausius}}(t) = S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \frac{t^2}{2} \int_{\mathcal{B}} \{T_{ab}(0) k_{\pm}^a k_{\pm}^b\} d^2\mathcal{A} + \mathcal{O}(t^3). \quad (80)$$

The constant term $S_{\mathcal{B}}$ is again undetermined by this argument. It is to be emphasized that this construction is to be applied to bifurcate null surfaces, not spacelike volumes, and is qualitatively different from entropy estimates built up by integrating up the thermodynamic entropy associated with small individual lumps of matter [29, 30, 31, 32]. If we choose to work with arbitrary affine parameters and arbitrary spacelike sections \mathcal{S} of the bifurcate null surface then we can rewrite the result as

$$S_{\text{Clausius}}(\mathcal{S}) = S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \int_{\mathcal{B}} \left\{ \frac{\lambda^2}{2} T_{ab}(0) k_{\pm}^a k_{\pm}^b \right\} d^2\mathcal{A} + \mathcal{O}(\lambda^3). \quad (81)$$

Here \mathcal{S} is now the 2-surface defined by propagating an affine distance λ along each null generator emanating from the bifurcation 2-surface \mathcal{B} .

4.2. General formula for curved-space Clausius entropy

In view of the above discussion we can now simply *postulate* that for arbitrary bifurcate null surfaces in curved spacetime, at arbitrary cross-section \mathcal{S} of the null surface

$$S_{\text{Clausius}}(\mathcal{S}) \equiv S_{\mathcal{B}} + \frac{2\pi k_B}{\hbar} \int_{\mathcal{B}}^{\mathcal{S}} \lambda T_{ab}(x(\xi) + \lambda k_{\pm}(\xi)) k_{\pm}^a k_{\pm}^b d^2\mathcal{A} d\lambda. \quad (82)$$

Note that λ is an affine null parameter, that this integral is well-defined in the sense that it is invariant under rescaling of the affine null parameter, and that in view of the preceding discussion this construction passes all the consistency tests one might reasonably wish to impose. Note in particular that imposing the classical null energy condition — the NEC — would guarantee positivity of the Clausius entropy flux. (While there are certainly quantum-induced violations of the energy conditions [33], we would argue that they can be neglected in the thermodynamic limit.) The only real restriction on the construction is that one should stop using it as soon as the null surface develops self-intersections.

5. Discussion

The net result of this calculation, and the construction it inspires, is that one can associate an observer-dependent notion of entropy, very closely related to the Clausius entropy [1], and a generalization of Jacobson’s local-Rindler entropy [27], to any arbitrary bifurcate null surface. That is, there is a certain sense in which even Clausius entropy is observer-dependent, with a “virtual Clausius entropy” being associated with arbitrary bifurcate “virtual causal horizons”. (See for instance [24, 25, 26].) This construction, because it generalizes one part of Jacobson’s “thermodynamic” derivation of the Einstein equations, cuts to the heart of the issue of the putative universal equality of thermodynamic and entanglement entropy. We will address such issues more fully in future work.

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